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On the common neighborhood graphs

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Abstract

Let G be a simple graph with vertex set $\{v_1, v_2, \ldots, v_n\}$. The common neighborhood graph (congraph) of G, denoted by $con(G)$, is a graph with vertex set $\{v_1, v_2, \ldots, v_n\}$, in which two vertices are adjacent if and only if they have at least one common neighbor in the graph G. In this paper we compute the common neighborhood of some composite graphs. In continue we investigate the relation between hamiltonicity of graph G and $con(G)$. Also we obtain a lower bound for the clique number of $con(G)$ in terms of clique number of graph G. Finally we state that the total chromatic number of G is bounded by chromatic number of $con(T(G))$.

Keywords: Common neighborhood graph, Hamiltonian cycle, Clique number, Graph operation, Chromatic number.

1 Introduction

In this paper, we only consider simple connected graphs. Let G and H be two graphs with vertex sets $V(G)$, $V(H)$ and edge sets $E(G)$, $E(H)$, respectively. For any vertex $v \in V(G)$, the set of neighbors of v is the set $N_G(v) = \{u \in V(G) | uv \in E(G)\}$. The Cartesian product of graphs G and H is denoted by

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$G \times H$. Also $G + H$ is the join of graphs G and H. The composition and tensor product of graphs G and H is denoted by $G[H]$ and $G \otimes H$, respectively. Definition of these graph operations is appeared in [7]. For given graphs G and H, their corona product, $G \circ H$ is obtained by taking $|V(G)|$ copies of H and joining each vertex of the i-th copy with vertex $v_i \in V(G)$.

A spanning cycle in a graph is a cycle that passes through each vertex exactly once. These are usually called hamiltonian cycles. A graph with this property is called hamiltonian. Let G be a simple graph with vertex set $\{v_1, v_2, \ldots, v_n\}$. The common neighborhood graph (congraph) of G, denoted by $con(G)$, is a graph with the vertex set $\{v_1, v_2, \ldots, v_n\}$, and two vertices are adjacent if and only if they have at least one common neighbor in the graph G [1,2].

A clique in a graph is a set of mutually adjacent vertices. The maximum size of a clique in a graph G is called the clique number of G and denoted by $\omega(G)$. A proper k-colouring, of a graph $G = (V, E)$ is an assignment of k colours to the vertices of G as no two adjacent vertices are assigned the same colour. The minimum k for which a graph G has a proper k-colouring is called its chromatic number, and denoted $\chi(G)$. A proper k-edge-colouring of a graph G is an assignment of k colours to the edges of G in which adjacent edges receive distinct colours. The edge chromatic number, $\chi^{'}(G)$, of a graph G is the minimum k for which G has k-edge-colouring. A total colouring of a graph G is a colouring $c : V \cup E \longrightarrow S$. The colouring c is proper if its restriction to V is a proper vertex colouring of G, its restriction to E is a proper edge colouring of G, and no edge receives the same colour as either of its ends. The total chromatic number of G, denoted by $\chi^{''}(G)$, is the minimum number of colours in a proper total colouring of G.

Our other notations are standard and taken mainly from [3,7]. Also in obtained results, graphs considered simple and if a graph has parallel edges, we consider only one edge of them.

2 Main Results

In this section we obtain $con(G)$ for different operations on two graphs. For more study on the graph operations, we refer the reader to study [5,6].

Theorem 2.1 *Let G_1 be a graph without isolated vertices, then $con(G_1 + G_2) = K_{n_1+n_2}$, where n_i, $1 \leq i \leq 2$, are the number of vertices in the graph G_i.*

Theorem 2.2 *Let G and H be two graphs. Then*

$$con(G \times H) = (con(G) \times con(H)) \cup E\,(G \otimes H)\,.$$

Theorem 2.3 *Let G and H be two graphs such that they have no isolated vertices. Then $con(G[H]) = con(G)[K_m] \cup E\left(G[\overline{K_m}]\right)$, where m is the number of vertices of the graph H and $\overline{K_m}$ is the complement of complete graph K_m.*

Theorem 2.4 *Let G and H be two graphs such that H has no isolated vertices. Then $con(G \circ H) = (con(G) \circ K_m) \cup E\left(\bigcup_{i=1}^{n} N_G(v_i) + H_i\right)$, where n and m are the number of vertices of the graphs G and H, respectively.*

Suppose G and H are two graphs with disjoint vertex sets. For given vertices $y \in V(G)$ and $z \in V(H)$ a splice of G and H by vertices y and z, $(G \cdot H)(y, z)$, is defined by identifying the vertices y and z in the union of G and H. Similarly, a link of G and H by vertices y and z is defined as the graph $(G \sim H)(y, z)$ obtained by joining y and z by an edge in the union of these graphs [4].

Theorem 2.5 *Let G and H be two graphs. Then*

$$con((G \cdot H)(v, u)) = (con(G) \cdot con(H))\,(v, u) \cup E\,(N_G(v) + N_H(u))\,,$$

$$\begin{aligned} con((G \sim H)(v, u)) &= (con(G) \sim con(H))\,(v, u) \\ &\cup E\,(v + N_H(u)) \cup E\,(u + N_G(v))\,, \end{aligned}$$

where $v + N_H(u)$, is the join of the vertex v and the neighbors of the vertex u in the graph H.

3 The relation between some special graphs and their common neighborhood graphs

In this section we compute the common neighborhood graph of subdivision graph, total graph and two extra subdivision-related graphs named $R(G)$ and $Q(G)$.

For a given graph G, the line graph of G is denoted by $L(G)$ and the vertices of $L(G)$ are the edges of G. Two edges of G that share a vertex are considered to be adjacent in $L(G)$. Subdivision graph of the graph G is denoted by $S(G)$ and is the graph obtained by inserting an additional vertex in each edge of G. Equivalently, each edge of G is replaced by a path of length 2. The total graph $T(G)$ of a graph G is a graph such that the vertex set of T corresponds to the vertices and edges of G and two vertices are adjacent in T if and only if their corresponding elements are either adjacent or incident in G. Two extra subdivision-related graphs are named $R(G)$ and $Q(G)$. In fact,

$R(G)$ is obtained from G by adding a new vertex corresponding to each edge of G, then joining each new vertex to the end vertices of the corresponding edge. Another way to describe $R(G)$ is to replace each edge of G by a triangle. Also $Q(G)$ is obtained from G by inserting a new vertex into each edge of G, then joining with edges those pairs of new vertices on adjacent edges of G.

Theorem 3.1 *Let G be a graph. Then $con(S(G)) = G \cup L(G)$.*

In continue $\{v_{e_1}, \ldots, v_{e_m}\}$ is the set of vertices correspond to the set of edges $\{e_1, \ldots, e_m\}$. Also $v_{e_i} + (N_G(v_k) \cup N_G(v_j))$ is the join of the vertex v_{e_i} and $N_G(v_k) \cup N_G(v_j)$.

Theorem 3.2 *Let G be a graph with the set of vertices $V(G) = \{v_1, \ldots, v_n\}$ and the set of edges $E(G) = \{e_1, \ldots, e_m\}$. We denote the vertices of $R(G)$ by $\{v_1, \ldots, v_n, v_{e_1}, \ldots, v_{e_m}\}$. Then*

$$con(R(G)) = G \cup L(G) \cup E(con(G)) \cup (\cup_{e_i = v_k v_j \in E(G)} v_{e_i} + (N_G(v_k) \cup N_G(v_j))).$$

Corollary 3.3 *Let G be a graph. Then $R(G)$ is the subgraph of $con(R(G))$.*

In the graph G, let $\{e_1, \ldots, e_k\}$ be all of the edges that have the vertex u as the common vertex. In continue we denote set of $\{v_{e_1}, \ldots, v_{e_k}\}$, by $N'_G(u)$.

Theorem 3.4 *Let G be a graph. Then*

$$con(Q(G)) = G \cup L(G) \cup (\cup_{e=uv \in E(G)} N'_G(u) + N'_G(v))$$
$$\cup (\cup_{e=uv \in E(G)} v_e + (N_G(u) \cup N_G(v))).$$

Corollary 3.5 *Let G be a graph. Then $Q(G)$ is the subgraph of $con(Q(G))$.*

Theorem 3.6 *Let G be a graph. Then*

$$con(T(G)) = G \cup E(con(G)) \cup L(G) \cup con(L(G))$$
$$\cup (\cup_{e=uv \in E(G)} v_e + (N_G(u) \cup N_G(v))).$$

Corollary 3.7 *Let G be a graph. Then $T(G)$ is the subgraph of $con(T(G))$.*

4 More results on common neighborhood graphs

In this section, we obtain some other results on common neighborhood graphs.

Theorem 4.1 *Let G be a connected graph with n vertices, $n \geq 3$ and n be an odd number. If G is a hamiltonian graph, then $con(G)$ is also hamiltonian graph.*

Note that the inverse of above theorem, does not hold. In Figure 1, T is a graph with odd number of vertices with no hamiltonian cycle, but $con(T)$ has a hamiltonian cycle $v_2v_5v_3v_1v_4v_2$.

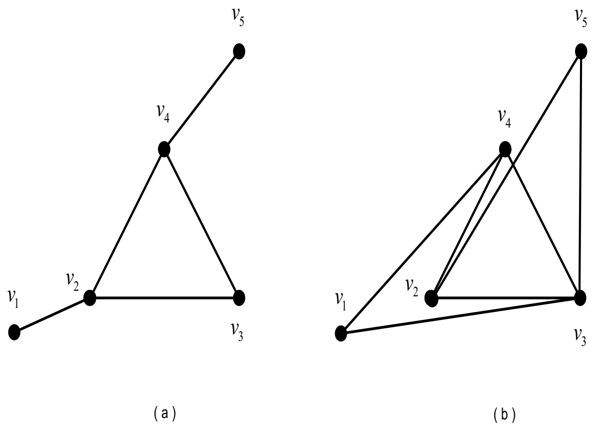

Figure 1. (a) A Nonhamiltonian Graph T, (b) The Hamiltonian Graph $con(T)$.

Also, if G is a hamiltonian graph with n vertices, in which n is an even number, then $con(G)$ is not necessary a hamiltonian graph. For example one can consider $K_{a,a}$. It is easy to see that $K_{a,a}$ is a hamiltonian graph and $con(K_{a,a}) = K_a \bigcup K_a$, but $K_a \bigcup K_a$ is a non-connected graph and without hamiltonian cycle.
In generally, if $con(G)$ is a hamiltonian graph, then it is not conclude that G is hamiltonian graph, for example one can see the Figure 2.

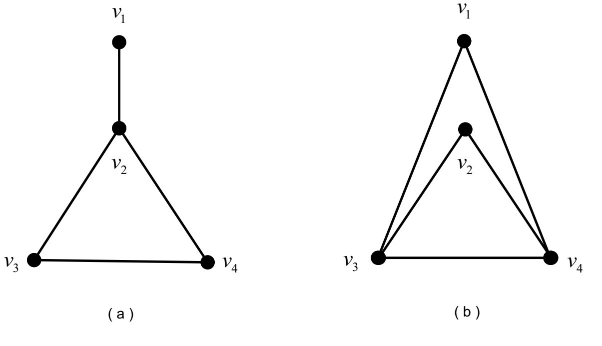

Figure 2. (a) A Nonhamiltonian Graph S, (b) The Hamiltonian Graph $con(S)$.

Theorem 4.2 *Let G be a connected graph, with $\omega(G) \geq 3$. Then $\omega(con(G)) \geq \omega(G)$.*

It is esay to see that if H is a subgraph of G and G has a proper k-colouring, then so is H. Thus $\chi(G) \geq \chi(H)$. Also by definition of total graph, it follows that $\chi^{''}(G) = \chi(T(G))$ for every graph G. Therefore by using Corollary 3.7., we have following corollary.

Corollary 4.3 *Let G be a graph. Then $\chi^{''}(G) \leq \chi(con(T(G)))$.*

5 Conclusions

In this paper we are computed the common neighborhood of some product graphs such as Cartesian product, join, composition and corona product. Also we obtained the common neighborhood graph of the splice and link of two graphs according to their common neighborhood graphs. In continue computed the common neighborhood graph of subdivision graph, total graph and two extra subdivision-related graphs that named $R(G)$ and $Q(G)$. Next the relation between hamiltonicity of graph G and $con(G)$ has been investigated. Also we gave a lower bound for the clique number of $con(G)$ in terms of clique number of graph G. Finally it is stated that total chromatic number of graph G is bounded by chromatic number of $con(T(G))$.

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